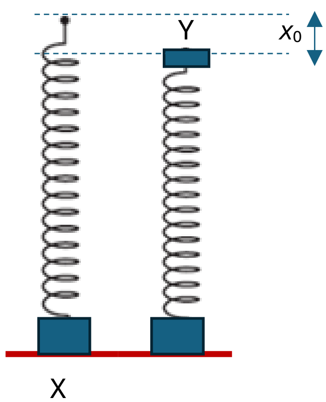
Teacher notes
Topic C

A challenging but instructive problem on SHM and much more.

A problem combining momentum and energy conservation along with details of simple harmonic motion and direction of forces.

Perfect for a cold, dark and rainy day!

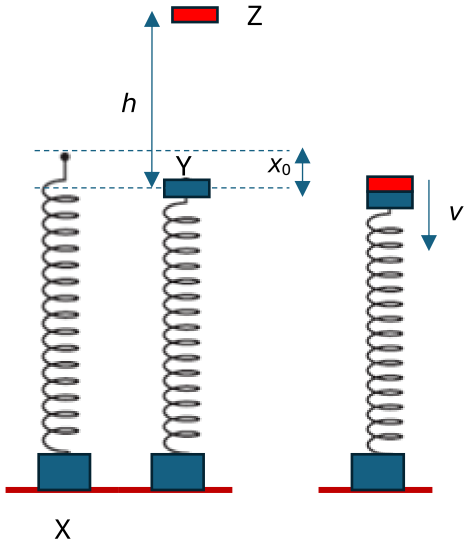
(a) A block X of mass 2*m* is attached to a vertical spring of spring constant *k*. A block Y of mass *m* is placed on top of the spring so that the spring is compressed by a distance x_0. The system is in equilibrium.



(i) Explain why $x_0 = \frac{mg}{k}$.

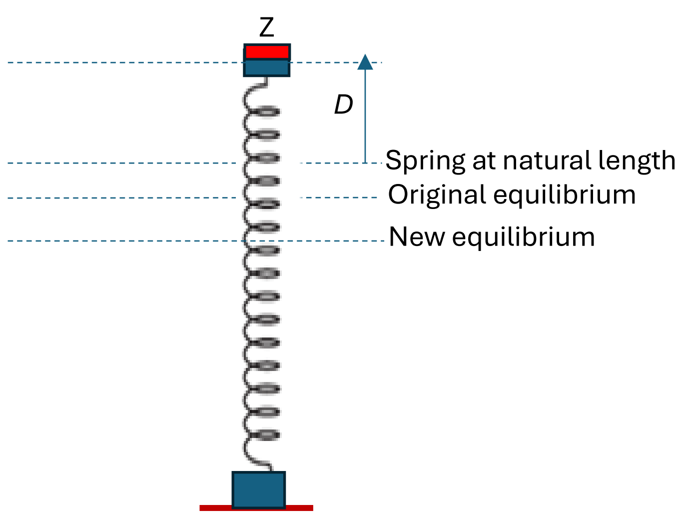
(ii) Determine the normal force on X from the ground.

(b) A third block Z of mass *m* is released from rest from a height *h* above Y. After the collision Z and Y move together without being stuck to each other.



(i) Determine, in terms of h, the speed v with which Y and Z begin to move, explaining your work.
(ii) Explain why the motion of the combined Y and Z is simple harmonic.
(iii) Determine the angular frequency of the oscillations.

(c) Y and Z perform simple harmonic oscillations. The largest displacement of Y and Z from the position where the spring has its natural length is D.



(i) Determine the largest value of D such that X never loses contact with the ground.
(ii) Hence determine the largest value of h for X not to lose contact with the ground.
(iii) Calculate the largest normal force on X during the oscillations for the value of D in (c)(i).

(d) Write down the equation giving the displacement of Y and Z.

(e) Confirm using the formula derived in (d) that the initial speed of Y and Z is what you found in (b)(i).

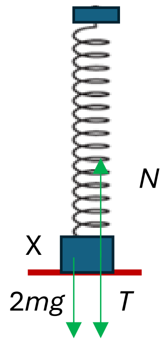
(f) Does Z lose contact with Y at any point during the oscillations?

Answers

(a)

(i) At equilibrium, $kx_0 = mg \Rightarrow x_0 = \dfrac{mg}{k}$.

(ii) The forces on X are:



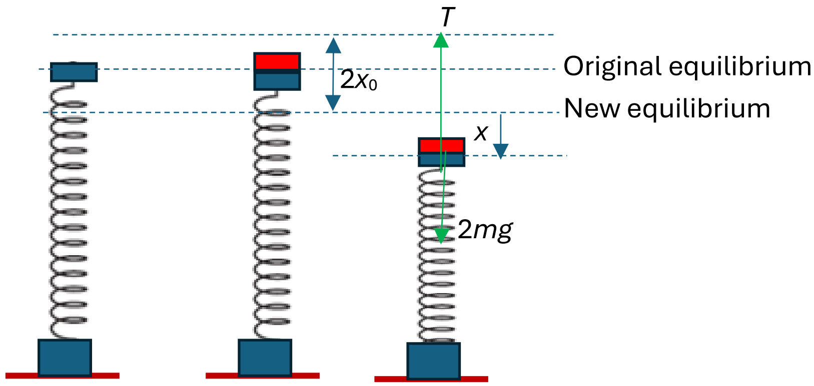
The tension force T is downward. The spring is compressed so the tension is upwards at the top end and downwards at the lower end. Hence,

$$N = 2mg + T = 2mg + kx_0 = 2mg + mg = 3mg.$$

(b)

(i) Z has speed $u = \sqrt{2gh}$, by energy conservation, when it hits Y. Applying conservation of momentum, $mu = 2mv \Rightarrow v = \dfrac{u}{2} = \dfrac{\sqrt{2gh}}{2}$. Momentum conservation is applicable since the external forces of weight and tension cancel out leaving a zero external net force on the system.

(ii) The diagram shows Y and Z in an arbitrary position a distance *x* below the new equilibrium position. The new equilibrium position is a distance x_0 below the original equilibrium position since the mass gets doubled: $(ke = 2mg \Rightarrow e = \dfrac{2mg}{k} = 2x_0)$.



The spring is compressed by $x+2x_0$. The net force on Y and Z is

$$T-2mg=k\left(x+2x_0\right)-2mg=kx+2kx_0-2mg=kx+2mg-2mg=kx$$

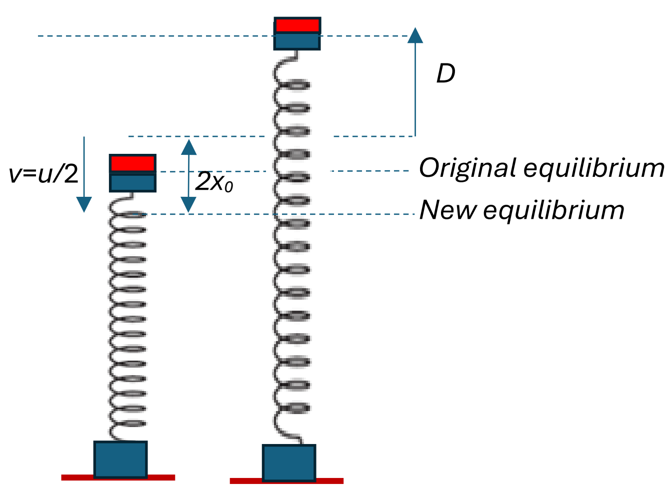
and is directed upwards i.e. opposite to *x*. The net force is opposite and proportional to the displacement from equilibrium and so SHM takes place.

(iii) From (ii), $2ma=-kx \Rightarrow a=-\frac{k}{2m}x$ which implies $\omega=\sqrt{\frac{k}{2m}}$.

(c)

(i) X will possibly lose contact with the ground when Y and Z are moving upwards. In that case the tension on Y and Z is downwards and so the tension on X is upwards. The net force on X just before X moves up is then $T+N-2mg=kD+N-2mg=0$. When X is about to lose contact, $N\to 0$ and so $kD=2mg \Rightarrow D=\frac{2mg}{k}$.

(ii) We apply energy conservation between the two positions shown below:



On the left we have kinetic energy and elastic energy: $E_T = \frac{1}{2}(2m)\left(\frac{u}{2}\right)^2 + \frac{1}{2}kx_0^2$

where *u* is the speed with which Z collides with Y. This can be written as

$$E_T = \frac{mu^2}{4} + \frac{1}{2}k\left(\frac{mg}{k}\right)^2 = \frac{mu^2}{4} + \frac{1}{2}\frac{(mg)^2}{k}.$$

On the right we have elastic energy and gravitational potential energy so

$$E_T = \frac{1}{2}kD^2 + 2mg\left(D + x_0\right) = \frac{1}{2}k\left(\frac{2mg}{k}\right)^2 + 2mg\frac{2mg}{k} + 2mg\frac{mg}{k} = \frac{2(mg)^2}{k} + \frac{4(mg)^2}{k} + \frac{2(mg)^2}{k} = \frac{8(mg)^2}{k}$$

.

Equating the two total energies:

$$\frac{mu^2}{4} + \frac{1}{2}\frac{(mg)^2}{k} = \frac{8(mg)^2}{k}$$

which gives

$$\frac{mu^2}{4} = \frac{15(mg)^2}{2k}$$

But $u^2 = 2gh$ so

$$\frac{m2gh}{4} = \frac{15(mg)^2}{2k} \quad \Rightarrow \quad h = \frac{15mg}{k}$$

(iii) The tension will be $T = k\left(D + 2x_0\right) = kD + 2kx_0 = 2mg + 2mg = 4mg$, directed downwards. The net force on X is $N - 2mg - T = N - 2mg - 4mg = N - 6mg$. Hence, *N* = 6*mg*.

(d) We take the up direction to be positive. The amplitude *A* of the motion is the largest displacement from equilibrium i.e. $A = D + 2x_0 = \frac{2mg}{k} + \frac{2mg}{k} = \frac{4mg}{k}$. The equation giving the displacement of Y and Z from the new equilibrium position as they perform SHM is therefore:

$$x = A\sin(\omega t + \phi) = \frac{4mg}{k}\sin(\omega t + \phi) \text{ with } \omega = \sqrt{\frac{k}{2m}}.$$

At *t* = 0 (Z impacts Y) we have that $x = x_0 = \frac{mg}{k}$. Hence

$$\frac{mg}{k} = \frac{4mg}{k}\sin(0 + \phi) \Rightarrow \sin\phi = \frac{1}{4}$$

Thus, $\phi = \sin^{-1}\frac{1}{4}$ *or* $\pi - \sin^{-1}\frac{1}{4}$.

The velocity at *t* = 0, is $v = \frac{4mg}{k}\omega\cos(0+\phi) = \frac{4mg}{k}\sqrt{\frac{k}{2m}}\cos\phi = g\sqrt{\frac{8m}{k}}\cos\phi$. The velocity is downward i.e. negative, and so we need a negative cosine. We must choose $\phi = \pi - \sin^{-1}\frac{1}{4}$. The value of the cosine is $\cos\phi = -\sqrt{1-\sin^2\phi} = -\sqrt{1-\frac{1}{16}} = -\frac{\sqrt{15}}{4}$.

The acceleration is given by

$$a = -\omega^2 x = -\frac{k}{2m}\frac{4mg}{k}\sin(\omega t + \phi) = -2g\sin(\omega t + \phi)$$

The maximum acceleration at the extremes of the oscillation is thus 2*g*.

(e) This velocity at *t* = 0 must equal $-\frac{\sqrt{2gh}}{2}$. Indeed, using $h = \frac{15mg}{k}$, we find $v = g\sqrt{\frac{8m}{k}}\cos\phi = -\sqrt{\frac{8gh}{15}}\frac{\sqrt{15}}{4} = -2\sqrt{2gh}\frac{1}{4} = -\frac{\sqrt{2gh}}{2}$ as it should be.

(f) When Y and Z are the highest point of the oscillation the acceleration of Y is 2*g* downwards. This means that Y will move down with acceleration 2*g* whereas Z will fall with acceleration *g*. Therefore, contact will be lost at the highest point.